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A. E. Glassgold

Lawrence Radiation Laboratory
University of California
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ABSTRACT

The nonspherically symmetric solutions to the Bardeen-Cooper-Schrieffer theory are given a physical interpretation in terms of an anisotropic fluid model. These solutions have been used previously to predict a phase transition in liquid by He^3 by Emery and Sessler and Anderson, Morel, Brueckner, and Soda. An investigation of the flow properties of such systems is made that involves the calculation of the effective mass for flow in a straight channel and the moment of inertia of a cylindrical container of the liquid. The angular-dependent energy-gap characteristic of this type of theory leads to an effective mass for flow that depends on the angle between the axis of symmetry of the fluid and the direction of flow. It also vanishes as the absolute temperature tends to zero, although not as rapidly as for a spherically symmetric gap. The moment of inertia, when the symmetry direction for the fluid and the rotation axis are the same, is simply related to the mass for flow.

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I. INTRODUCTION

The work of Bardeen, Cooper, and Schrieffer (BCS) provides a remarkably successful solution to the problem of superconductivity.[1] The basic feature in their approach is the strong correlation between conduction electrons with equal and opposite momentum and spin. This type of correlation probably plays an essential role in other many-fermion systems. For example, Van Hove has shown how the usual perturbation theory for an imperfect Fermi gas breaks down under just those conditions when the BCS approach is valid.[2]

Direct extensions of the BCS theory have already been made to finite nuclei,[3] infinite nuclear matter, and liquid He^3.[4,5] Of special interest is the prediction that liquid He^3 undergoes a phase transition at very low temperatures to a highly correlated phase similar to the phase change observed for superconductors.[6,7] The predicted transition temperature is of the order of 0.07° K, but so far no anomalous effects have been observed just above this temperature.[8]

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The theoretical description of this phase transition differs from that for the electrons in a superconductor in the following important respect. If the Fermi surface in a metal is considered to be spherically symmetric, then the correlation function in the original BCS theory is spherically symmetric. For liquid He^3, on the other hand, the correlation function is not thought to be spherically symmetric. (This is a direct consequence of the fact that the interaction at the Fermi surface for two helium atoms in a relative S state is repulsive.) The possible existence of such solutions in the BCS theory was first noted by Anderson.[9] The anisotropic correlations contained in these solutions raise interesting questions of interpretation, particularly for liquid He^3, where there is no long-range order.

It is the purpose of this paper to discuss the physical significance of these anisotropic solutions in the BCS theory. We often consider liquid He^3 as a specific example, although much of the discussion is more general. The interpretation is mainly given in terms of two quantities, the <u>effective mass for flow</u> through a straight channel, and the <u>moment of inertia</u> for the rotation of a cylindrical container of the fluid. These quantities determine the ability of the fluid to transport linear and angular momentum.

Before the effective masses for flow and rotation are calculated in Sections IV and V, Bogolyubov's "quasi-particle" form of the BCS theory is reviewed in Section II. The physical interpretation of the theory in terms of an anisotropic fluid is also given in this section. In Section III the general formulae for the inertial parameters are reviewed.

II. QUASI-PARTICLE THEORY OF SUPERFLUID FERMIONS SYSTEMS

Bogolyubov has emphasized the quasi-particle nature of the BCS theory.[5] By a quasi-particle approximation, we mean that the actual Hamiltonian for this problem is truncated and transformed into the form

$$H_0 = E_0 + \sum_{\underset{\sim}{k}} E(\underset{\sim}{k})(\alpha_{\underset{\sim}{k}}^\dagger \alpha_{\underset{\sim}{k}} + \beta_{\underset{\sim}{k}}^\dagger \beta_{\underset{\sim}{k}}) . \tag{2.1}$$

The operators $\alpha_{\underset{\sim}{k}}^\dagger$ and $\beta_{\underset{\sim}{k}}^\dagger$ ($\alpha_{\underset{\sim}{k}}$ and $\beta_{\underset{\sim}{k}}$) create (destroy) the excitations of the many-particle system. These excitations have definite energy $E(\underset{\sim}{k})$ and momentum $\underset{\sim}{k}$. The quasi-particle operators obey the same anti-commutation rules as the corresponding operators for the actual particles making up the system. (In order to avoid introducing a spin label, we use two sets of quasi-particle operators.) The linear transformation between particle operators and quasi-particle operators is

$$\begin{aligned} \alpha_{\underset{\sim}{k}} &= u(\underset{\sim}{k})\, a_{\underset{\sim}{k}+} - v(\underset{\sim}{k})\, a^\dagger_{-\underset{\sim}{k}-} , \\ \beta_{\underset{\sim}{k}} &= u(\underset{\sim}{k})\, a_{-\underset{\sim}{k}-} + v(\underset{\sim}{k})\, a^\dagger_{\underset{\sim}{k}+} \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} a_{\underset{\sim}{k}+} &= u(\underset{\sim}{k})^*\, \alpha_{\underset{\sim}{k}} + v(\underset{\sim}{k})\, \beta^\dagger_{\underset{\sim}{k}} , \\ a_{\underset{\sim}{k}-} &= u(\underset{\sim}{k})^*\, \beta_{-\underset{\sim}{k}} - v(\underset{\sim}{k})\, \alpha^\dagger_{-\underset{\sim}{k}} \end{aligned} \tag{2.2a}$$

The operators $a^{\dagger}_{\underset{\sim}{k}\sigma}$ $(a_{\underset{\sim}{k}\sigma})$ create free-particle states of momentum $\underset{\sim}{k}$ and "spin" projection $\sigma = \pm 1$.[10] The anticommutation relations are preserved for

$$|u(\underset{\sim}{k})|^2 + |v(\underset{\sim}{k})|^2 = 1 . \tag{2.3}$$

It has also been assumed that we have $u(-\underset{\sim}{k}) = u(\underset{\sim}{k})$ and $v(-\underset{\sim}{k}) = v(\underset{\sim}{k})$.

According to Eq. (2.3) we may write the two complex functions as

$$\begin{cases} u(\underset{\sim}{k}) = \cos \chi(\underset{\sim}{k})\, e^{i\eta(\underset{\sim}{k})} \\ v(\underset{\sim}{k}) = \sin \chi(\underset{\sim}{k})\, e^{i\zeta(\underset{\sim}{k})} . \end{cases} \tag{2.4}$$

It can be shown that all physical observables depend only on the difference in phase,

$$\phi(\underset{\sim}{k}) = \zeta(\underset{\sim}{k}) - \eta(\underset{\sim}{k}) . \tag{2.5}$$

Hence the two real functions, $\chi(\underset{\sim}{k})$ and $\phi(\underset{\sim}{k})$, characterize the quasi-particle transformation. At absolute zero, Bogolyubov determined the transformation in the following way. The Hamiltonian of the system is written in the new representation with all creation operators to the left. No quadrilinear terms are retained and the resulting <u>truncated</u> Hamiltonian is diagonalized, i.e. forced to have the form of Eq. (2.1). This procedure is equivalent to the BCS variational calculation of the ground-state energy. At finite temperatures, the thermodynamic potential is minimized instead (as discussed, for example, in Reference 6). As a result, the theory is essentially determined by the following coupled equations:

$$C(\underset{\sim}{k}) = -\frac{1}{2}\sum_{\underset{\sim}{k}'}(\underset{\sim}{k} - \underset{\sim}{k}\,|\,v\,|\,\underset{\sim}{k}' - \underset{\sim}{k}')\,\frac{C(\underset{\sim}{k}')}{E(\underset{\sim}{k}')}\,\tanh\,\frac{1}{2}\,\beta\,E(\underset{\sim}{k}')\,, \tag{2.6}$$

$$\xi(\underset{\sim}{k}) = [\epsilon(\underset{\sim}{k}) - \mu] + \sum_{\underset{\sim}{k}'}(\underset{\sim}{k}\,\underset{\sim}{k}'\,|\,\overline{v}\,|\,\underset{\sim}{k}\,\underset{\sim}{k}')\,\{f(\underset{\sim}{k}') + [1 - 2f(\underset{\sim}{k}')]\,|\,v(\underset{\sim}{k}')\,|^2\} \tag{2.7}$$

The function C is defined as

$$C(\underset{\sim}{k}) = \sum_{\underset{\sim}{k}'}(\underset{\sim}{k} - \underset{\sim}{k}\,|\,v\,|\,\underset{\sim}{k}' - \underset{\sim}{k}')\,u^*(\underset{\sim}{k}')\,v(\underset{\sim}{k}')\,[1 - 2f(\underset{\sim}{k}')]\,, \tag{2.8}$$

where

$$E(\underset{\sim}{k}) = \sqrt{\xi^2(\underset{\sim}{k}) + |\,C(\underset{\sim}{k})\,|^2} \tag{2.9}$$

and

$$f(\underset{\sim}{k}) = \frac{1}{e^{\beta E(\underset{\sim}{k})} + 1} \tag{2.10}$$

The symbol μ stands for the chemical potential and $\epsilon(\underset{\sim}{k})$ for the unperturbed single-particle energy. For a spherically symmetric Fermi surface ϵ depends only on the magnitude $k = |\underset{\sim}{k}|$. The matrix elements of the two-body potential are $(\underset{\sim}{k}_1\,\underset{\sim}{k}_2\,|\,v\,|\,\underset{\sim}{k}_1'\,\underset{\sim}{k}_2')$; the forward scattering of the quasiparticles, which appears in the expression for their energy in Eq. (2.7), is

$$(\underset{\sim}{k}\,\underset{\sim}{k}'\,|\,\bar{v}\,|\,\underset{\sim}{k}\,\underset{\sim}{k}') = (\underset{\sim}{k}\,\underset{\sim}{k}'\,|\,v\,|\,\underset{\sim}{k}\,\underset{\sim}{k}') - (\underset{\sim}{k}\,\underset{\sim}{k}'\,|\,v\,|\,\underset{\sim}{k}'\,\underset{\sim}{k}) + (\underset{\sim}{k} - \underset{\sim}{k}'\,|\,v\,|\,\underset{\sim}{k} - \underset{\sim}{k}'). \tag{2.11}$$

We also note

$$C(\underset{\sim}{k}) = |\,C(\underset{\sim}{k})\,|\,e^{i\phi(\underset{\sim}{k})}, \tag{2.12}$$

where $\phi(\underset{\sim}{k})$ was defined in Eq. (2.5), and

$$\tan 2\chi(\underset{\sim}{k}) = -\frac{|C(\underset{\sim}{k})|}{\xi(\underset{\sim}{k})}. \tag{2.13}$$

In this brief resume of the theory, we have indicated explicitly the possible dependence of the properties of an excitation on its vector momentum, in particular, on its direction measured with respect to an arbitrary axis $\hat{n}$ henceforth called the "quantization" axis. [The original BCS theory of superconductivity for a spherically-symmetric Fermi surface corresponds to the special case of isotropic properties.] The possibility of this anisotropy stems directly from the lack of invariance of the truncated Hamiltonian under an arbitrary rotation, which in turn arises from the direction dependence of the excitation energies in Eq. (2.1). This absence of rotational symmetry is due to the truncation process, since the original many-particle Hamiltonian describing the liquid is certainly invariant under arbitrary rotations. (It should be noted that the quasi-particle transformation of the original Hamiltonian leads to a new Hamiltonian that is still rotation-invariant. This is true even for the angular-dependent solutions, since Eq. (2.3), the requirement that the transformation be canonical, is satisfied.)

Despite the fact that the model Hamiltonian is not invariant under arbitrary rotations, there are physical situations to which the solutions correspond. For example, at absolute zero, the ground state corresponds to a fluid with a preferred direction common to the whole sample and determined by the walls of the container. In this case, the arbitrarily small interactions with the walls (which are not usually included in the original rotationally invariant Hamiltonian) play a crucial role just as in the formation of a crystal. Other cases in which the walls serve to establish preferred directions are quasi-equilibrium situations corresponding to macroscopic fluid flow, discussed more fully in the next sections.

To arrive at a better understanding of the quasi-particle model with angular-dependent solutions, we recall that the quantity $C(\underline{k})$ determines the pair-correlation function. The pair-correlation function in this type of theory describes short-range order, with a correlation length of order $\beta_c \hbar v_F$ (where v_F is the Fermi velocity and $\beta_c{}^{-1}$ is the transition temperature). In addition, the particle density is uniform and isotropic, whereas the correlation function is angular-dependent. In other words, we are describing here an anisotropic liquid.[11]

The correlation length in the BCS theory is rather large compared with atomic spacings. For example, for He^3, for which the transition temperature is predicted to be of the order of 0.07° K, the correlation length is about 100 Å. For equilibrium at a nonzero temperature, this implies the formation of a loose domain structure with a domain size no smaller than the correlation length. The existence of a domain structure for this system was suggested by Anderson et al.[7] When the pair-correlation function is anisotropic, each domain has a preferred axis and, in first approximation, these domains are randomly oriented.

The existence of domains is inferred from the following energetic considerations. Particles in the liquid interact strongly only if they are within a correlation length of one another. Therefore the division of a domain in two has associated with it an increase in the total energy of the system which is proportional to the correlation length times the surface area in contact. Thus a negligible change in the total energy of the sample is required for the sample to break up into a large number of domains. At a nonzero temperature the number of domains into which the fluid is subdivided is determined by the condition that the formation energy of a domain is of the order of kT. As a consequence, at absolute zero, there is just a single domain, as was previously remarked. On the other hand, as the transition temperature is approached from below, the number of domains increases rapidly, since the correlation energy approaches zero. For quasi-equilibrium situations corresponding to fluid flow, these energetic considerations must be extended; this is done in the following sections.

III. GENERAL FORMULAE FOR THE INERTIAL PROPERTIES OF A SUPERFLUID

We now discuss the superfluid properties of the system in a quantitative way, using the effective masses for uniform translation and rotation. Our discussion is motivated by Landau's discussion of the superfluidity of liquid He II.[12] For the special case of spherically symmetric solutions, Bardeen[13] and Khalatnikov and Abrikosov[14] have already discussed the relation between the BCS theory and the two-fluid model. These authors have calculated the density of normal electrons, which is simply proportional to our effective mass for flow. In this section we review the general statistical formulae for the inertial parameters. The explicit calculation of the effective mass for flow and the moment of inertia is discussed separately in succeeding sections.

1. Effective Mass for Flow

We consider the uniform flow of the fluid down an infinite channel. If $\underset{\sim}{v}$ is the mean drift velocity of the excitations and if $\langle \underset{\sim}{P} \rangle$ is the mean total momentum per unit volume, then the effective mass for flow is defined by the equation

$$\langle \underset{\sim}{P} \rangle = M_f (\underset{\sim}{v}) \underset{\sim}{v} . \tag{3.1}$$

The velocity $\underset{\sim}{v}$ is, by definition, the velocity (with respect to the laboratory system) of the reference frame in which the quasi-particle distribution function is that for a fluid at rest, i.e. Eq. (2.10) for this problem. Unless stated otherwise, the effective mass for flow is that obtained in the limit of zero velocity,

$$M_f(0) = \left. \frac{\partial \underset{\sim}{P} (\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right|_{\underset{\sim}{v} = 0} \tag{3.2}$$

We conveniently define a superfluid as a system with $M_f(0) < nm$, where n is the density and m the particle mass. This definition of a superfluid emphasizes the contrast with a classical fluid with respect to a liquid's ability to transfer momentum. We note that Landau's normal density is just $\rho_n = M_f (0) /m$.

According to the general principles of statistical mechanics, the mean momentum per unit volume is

$$\langle \underset{\sim}{P} \rangle = \frac{\mathrm{Tr}\,[\,\underset{\sim}{P}\, e^{-\beta(H - \mu N - \underset{\sim}{P} \cdot \underset{\sim}{v})}\,]}{\mathrm{Tr}\,[\, e^{-\beta(H - \mu N - \underset{\sim}{P} \cdot \underset{\sim}{v})}\,]} \tag{3.3}$$

The symbol $\mathrm{Tr}\,[\cdot \cdot \cdot]$ indicates the trace operation appropriate to the grand canonical ensemble, and H, N, and $\underset{\sim}{P}$ are the operators for the Hamiltonian, the number of particles, and momentum density, respectively. Carrying out the differentiation indicated in Eq. (3.2), using the fact $\underset{\sim}{P}$ commutes with $H - \mu N$, and that $\langle \underset{\sim}{P} \rangle$ is 0 for $\underset{\sim}{v} = 0$, we obtain the formula for the effective mass for flow:

$$M_f(0) = \beta \left\langle (\underset{\sim}{P} \cdot \hat{\underset{\sim}{v}})^2 \right\rangle , \tag{3.4}$$

where $\hat{\underset{\sim}{v}} = \underset{\sim}{v} / v$. The statistical average is carried out in the rest frame ($\underset{\sim}{v} = 0$). We emphasize once more that this is just Landau's definition of the normal density.

2. Moment of Inertia

We now consider a cylindrical container of the fluid rotating with angular velocity ω about its axis of symmetry $\hat{\omega}$. If $\underset{\sim}{J}$ is the operator for the total angular momentum of the system, then the moment of inertia is defined by the relation

$$\left\langle \underset{\sim}{L} \cdot \hat{\underset{\sim}{\omega}} \right\rangle = I(\omega)\omega . \tag{3.5}$$

We discuss only the limiting value

$$I(0) = \frac{\partial}{\partial \omega} \left\langle \underset{\sim}{L} \cdot \hat{\underset{\sim}{\omega}} \right\rangle \Big|_{\omega = 0} \tag{3.6}$$

By applying the same statistical equilibrium discussion used above for M_f, the formula for the moment of inertia is found to be

$$I(0) = \beta \left\langle (\underset{\sim}{J} \cdot \hat{\underset{\sim}{\omega}})^2 \right\rangle . \tag{3.7}$$

Again, the statistical average is carried out. This result is given by Blatt, Butler, and Schafroth.[15]

IV. EFFECTIVE MASS FOR FLOW

The above formulae, Eqs. (3.2) and (3.7), show how $M_f(0)$ and $I(0)$ are related to the statistical average of $(\underset{\sim}{P} \cdot \hat{\underset{\sim}{v}})^2$ and $(\underset{\sim}{J} \cdot \hat{\underset{\sim}{\omega}})^2$. The evaluation of these averages is carried out in the quasi-particle representation. This is exactly the procedure followed in a recent discussion of the moment of inertia for the low-density theory of liquid He^4.[16]

In order to evaluate Eq. (3.4) for $M_f(0)$, we need the expression for the momentum operator in the quasi-particle representation

$$\underset{\sim}{P} = \sum_{\underset{\sim}{k}} \underset{\sim}{k} \, (\alpha_{\underset{\sim}{k}}^{\dagger} \alpha_{\underset{\sim}{k}} - \beta_{\underset{\sim}{k}}^{\dagger} \beta_{\underset{\sim}{k}}) . \tag{4.1}$$

We next write the average of $(\underset{\sim}{P} \cdot \hat{\underset{\sim}{v}})^2$ as

$$\begin{aligned} \left\langle (\underset{\sim}{P} \cdot \hat{\underset{\sim}{v}})^2 \right\rangle = & \sum_{\underset{\sim}{k} \neq \underset{\sim}{k}'} (\underset{\sim}{k} \cdot \hat{\underset{\sim}{v}})(\underset{\sim}{k}' \cdot \hat{\underset{\sim}{v}}) \left(\left\langle \alpha_{\underset{\sim}{k}}^{\dagger} \alpha_{\underset{\sim}{k}} \right\rangle - \left\langle \beta_{\underset{\sim}{k}}^{\dagger} \beta_{\underset{\sim}{k}} \right\rangle \right) \left(\left\langle \alpha_{\underset{\sim}{k}'}^{\dagger} \alpha_{\underset{\sim}{k}'} \right\rangle - \left\langle \beta_{\underset{\sim}{k}'}^{\dagger} \beta_{\underset{\sim}{k}'} \right\rangle \right) \\ & + \sum_{\underset{\sim}{k}} (\underset{\sim}{k} \cdot \hat{\underset{\sim}{v}})^2 \left(\left\langle \alpha_{\underset{\sim}{k}}^{\dagger} \alpha_{\underset{\sim}{k}} \alpha_{\underset{\sim}{k}}^{\dagger} \alpha_{\underset{\sim}{k}} \right\rangle + \left\langle \beta_{\underset{\sim}{k}}^{\dagger} \beta_{\underset{\sim}{k}} \beta_{\underset{\sim}{k}}^{\dagger} \beta_{\underset{\sim}{k}} \right\rangle - 2 \left\langle \alpha_{\underset{\sim}{k}}^{\dagger} \alpha_{\underset{\sim}{k}} \right\rangle \left\langle \beta_{\underset{\sim}{k}}^{\dagger} \beta_{\underset{\sim}{k}} \right\rangle \right) . \end{aligned} \tag{4.2}$$

Since the statistical averages of $\alpha_{\underset{\sim}{k}}^{\dagger} \alpha_{\underset{\sim}{k}}$, $\beta_{\underset{\sim}{k}}^{\dagger} \beta_{\underset{\sim}{k}}$, and their squares are <u>all</u> just $f(\underset{\sim}{k})$, the first line of Eq. (4.2) is zero and the second line leads to the following equation for $M_f(0)$:

$$M_f(0) = 2\beta \sum_{\underset{\sim}{k}} (\underset{\sim}{k} \cdot v)^2 f(\underset{\sim}{k}) [1 - f(\underset{\sim}{k})] . \tag{4.3}$$

As remarked previously, this is essentially Landau's expression for the normal density.[12] This formula shows explicitly how the excitation spectrum, through the statistical factor $f(\underset{\sim}{k})$ determines the effective mass for flow.

For a spherically symmetric energy gap, $C(\underset{\sim}{k}) = \Delta$, corresponding to the "excitation spectrum"

$$E(k) = \sqrt{\left(\frac{k^2 - k_F^2}{2m}\right)^2 + \Delta^2} , \tag{4.4}$$

Eq. (4.3) becomes

$$\frac{M_f(0)}{n\,m} = 2\beta \int_{\Delta}^{\infty} dE \frac{E}{\sqrt{E^2 - \Delta^2}} \frac{e^{\beta E}}{(e^{\beta E} + 1)^2} . \tag{4.5}$$

The most important contributions of the integrand come from the neighborhood of the Fermi surface where $E = \Delta$. It is convenient to rewrite this equation as

$$\frac{M_f(0)}{nm} = 2 \int_0^\infty dx \, \frac{x + \lambda}{\sqrt{x^2 + 2\lambda x}} \, \frac{e^{x+\lambda}}{(e^{x+\lambda} + 1)^2} , \tag{4.6}$$

where $\lambda = \beta \Delta (\beta)$. It is now easy to establish the following asymptotic limits of this integral, corresponding to the limits $T \to 0$ ($\Delta \to \Delta_0$) and $T \to T_c$ ($\Delta \to 0$):

$$\frac{M_f}{nm} = \begin{cases} \sqrt{2\pi\lambda}\, e^{-\lambda} & \lambda \to \infty \; (T \to 0, \; \Delta \to \Delta_0) \qquad (4.7a) \\ 1 & \lambda \to 0 \; (T \to T_c , \; \Delta \to 0). \qquad (4.7b) \end{cases}$$

A more detailed discussion of M_f at intermediate temperature is given by Khalatnikov and Abrikosov.[14] As T decreases from T_c, M_f decreases (linearly at first) to zero, vanishing exponentially as absolute zero is approached. If the energy gap is set equal to zero for all temperatures, the case of the ideal gas is recovered. From Eq. (4.7b) we see that the effective mass for the flow of an ideal gas is the true mass.

For asymmetric solutions, the angular-dependent factor $(\underset{\sim}{k} \cdot \hat{\underset{\sim}{v}})^2$ in Eq. (4.3) is now important. We introduce the spherical polar coordinates (k, θ, ϕ) for the quasi-particle momentum $\underset{\sim}{k}$, with the preferred direction $\hat{n}$ of the domain under consideration as quantization axis, and the angle τ between $\hat{n}$ and $\hat{v}$. We assume here that the excitation spectrum has cylindrical symmetry about $\hat{n}$: $E = E(k, \theta)$; and, for simplicity, that

$C = C(\theta)$. In this case, Eq. (4.5) must be replaced by

$$\frac{M_f(0)}{n\,m} = 2\beta \int_{-1}^{1} d\,(\cos\theta)\ \frac{3}{2}\ (\cos^2\tau\cos^2\theta + \frac{1}{2}\sin^2\tau\sin^2\theta)$$

$$\times \int_{|C(\theta)|}^{\infty} d\,E\ \frac{E}{\sqrt{E^2 - |C(\theta)|^2}}\ \frac{e^{\beta E}}{(e^{\beta E}+1)^2}\ . \tag{4.9}$$

Current applications to liquid He^3 make use of the form

$$C(\underset{\sim}{k}) = \Delta_{\ell m}(\theta)\ Y_{\ell m}(\theta,\phi)\ , \tag{4.10}$$

and thus

$$|C(\underset{\sim}{k})|^2 = \Delta^2_{\ell m}(\theta)\ P_{\ell m}{}^2(\theta)\ .$$

This function vanishes at several points, and the contributions to the integrand of Eq. (4.9) from the neighborhood of these points are the most important ones. As a result, M_f does not vanish as rapaidly as $T \to 0$, as it does for a spherically symmetric gap.

We now turn our attention to the question of the orientation of the preferred axis $\hat{n}$ with respect to the flow direction $\hat{v}$ in an actual experiment. Equation (4.9) may be rewritten

$$\frac{M_f(0)}{n\,m} = \cos^2\tau\ K_1 + \sin^2\tau\ K_2\ , \tag{4.11}$$

where $(x = \cos\theta)$

$$K_1 = \int_{-1}^{1} dx \; x^2 \, F(x) \, , \tag{4.12a}$$

$$K_2 = \frac{1}{2} \int_{-1}^{1} dx \; (1 - x^2) \, F(x) \, , \tag{4.12b}$$

and

$$F(x) = 3\,\beta \int_{|C(x)|}^{\infty} dE \; \frac{E}{\sqrt{E^2 - |C(x)|^2}} \; \frac{e^{\beta E}}{(e^{\beta E} + 1)^2} \, . \tag{4.12c}$$

The mass for flow, and therefore the total energy, is a minimum for $\tau = 0$ and π, or $\tau = \pi/2$, depending on whether $K_2 > K_1$ or $K_1 > K_2$ holds.

In the special case, $K_1 = K_2$, the effective mass is independent of τ and all directions of the preferred axes are equally probable, energetically. In this improbable case ($K_1 = K_2$), the fluid would maintain its domain structure although the orientation of the various domain axes would be essentially uncorrelated. In the more likely situation, with $K_1 \neq K_2$, the preferred axes and the flow direction are, on the average, either perpendicular ($K_2 > K_1$) or parallel ($K_1 < K_2$). (There is no difference between $\tau = 0$ and $\tau = \pi$). There is, of course, a statistical distribution of the directions about these average values. Which of the two directions is most probable depends on the relative magnitude of K_1 and K_2. It is difficult to make a general conclusion on this point without obtaining more complete solutions to the basic equations (Eqs. (2.6) and (2.7)).

The above question can, of course, be discussed in the approximation of Eq. (4.10).[6,7] As $T \to 0$, the different energy gaps $\Delta_{\ell m}$ for the various m values are generally distinct, and the lowest energy is obtained with the largest energy gap. The integrals K_1 and K_2 can then be evaluated for this value of m and the parallel and perpendicular directions distinguished. For example, the solution that give the lowest energy for $\ell = 1$ is $C = \Delta_{11} \; Y_{11}$, and a simple calculation gives $K_1 > K_2$. This means that the preferred direction in the fluid is perpendicular to the flow direction in this case. As the temperature is increased, the fluid breaks up into domains, and there are Boltzmann distributions both for the domain directions and for the various solutions characterized by the different m values.

V. MOMENT OF INERTIA

Before evaluating Eq. (3.7) for the moment of inertia, we recall that, in the derivation of this equation, it is assumed that $(H - \mu N)$ and $\underset{\sim}{J} \cdot \hat{\underset{\sim}{\omega}}$ commute. Since $\underset{\sim}{J} \cdot \hat{\omega}$ is the projection of the total angular momentum along the axis of rotation, it follows that the operator $H - \mu N$ must be invariant under rotations about $\hat{\underset{\sim}{\omega}}$. This condition is fulfilled for quasi-particles whose excitation energy does not depend on ϕ, where k, θ, and ϕ are the spherical coordinates of the quasi-particle momentum $\underset{\sim}{k}$ with $\hat{\underset{\sim}{\omega}}$ as polar axis. This property is possessed by the approximate solutions to Eq. (2.6) given in Eq. (4.10), which are valid just below the transition temperature. There is a wider class of functions that vary as $e^{im\phi}$ and which, therefore, correspond to an axially symmetric model Hamiltonian. Since little is known about the general properties of the solutions to Eqs. (2.6) and (2.7), however, we cannot exclude even more general solutions.

In any case, the calculation of the moment of inertia in this section is confined to axially symmetric solutions for which the general formulae, Eq. (3.7), is valid. This corresponds to the physical situation in which there is a single preferred direction in the fluid parallel to the axis of rotation.

We now evaluate Eq. (3.7) for the moment of inertia following the method recently used for the low-density theory of liquid He^4.[16] The operator for the projection of the total angular momentum along the rotation axis is, in the notation of second quantization,

$$\underset{\sim}{J} \cdot \hat{\underset{\sim}{\omega}} = \sum_{\underset{\sim}{k}\,\underset{\sim}{k}'} \sum_{\sigma} L_{\underset{\sim}{k}\underset{\sim}{k}'} \, a^{\dagger}_{\underset{\sim}{k}\sigma} \, a_{\underset{\sim}{k}'\sigma} \,. \tag{5.1}$$

We ignore the negligible contribution of the intrinsic spin of the particles. The symbol L stands for the projection of the orbital angular momentum of one particle along $\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{\omega}}$. Its matrix elements in momentum space satisfy the relations

$$L_{\underset{\sim}{k}\underset{\sim}{k}'} = L^{*}_{\underset{\sim}{k}'\underset{\sim}{k}} \,, \tag{5.2a}$$

$$L_{\underset{\sim}{k}\underset{\sim}{k}'} = L_{-\underset{\sim}{k}-\underset{\sim}{k}'} \,, \tag{5.2b}$$

$$L_{\underset{\sim}{k}\underset{\sim}{k}'} = - L^{*}_{\underset{\sim}{k}\underset{\sim}{k}'} \,, \tag{5.2c}$$

and

$$L_{\underset{\sim}{k}\underset{\sim}{k}'} = L_{\underset{\sim}{k}\underset{\sim}{k}'} \, \delta_{k,k'} \, \delta_{\underset{\sim}{k}\cdot\hat{\underset{\sim}{n}},\, \underset{\sim}{k}'\cdot\hat{\underset{\sim}{n}}} \,, \tag{5.3a}$$

$$L_{\underset{\sim}{k}\underset{\sim}{k}} = 0 \,. \tag{5.3b}$$

Equations (5.2a), (5.2b), and (5.2c) follow from the requirements of hermiticity, inversion invariance, and time-reversal invariance. The last relation, Eq. (5.3), expresses the property of L as the generator of infinitesimal rotations about $\hat{n}$. Upon transformation to the quasi-particle representation by direct substitution of Eq. (2.1), Eq. (5.1) becomes

$$J \cdot \hat{n} = \sum_{kk'} L_{kk'} [u(k)\, u^*(k') + v(k)\, v^*(k')]\, (\alpha_k^\dagger\, \alpha_{k'} + \beta_k^\dagger\, \beta_{k'}) \,. \tag{5.4}$$

We note that $J \cdot \hat{n}$ involves only "diagonal operators," i.e., operators involving the same number of creation and destruction operators. That no other operators occur (such as products of two creation or two destruction operators) is a direct consequence of the axial symmetry of the quasi-particle transformation. Another consequence of this symmetry is

$$J \cdot \hat{n}\ |\, 0 > = 0 \,,$$

where $|\, 0 >$ is the ground-state or quasi-particle vacuum. Furthermore, the expectation value or the ensemble average of $J \cdot \hat{n}$ is <u>always</u> zero, since it involves the terms in Eq. (5.4) for which $k = k'$ and $L_{kk'} = 0$.

The square of $J \cdot \hat{n}$ which appears in Eq. (3.7) is

$$(J \cdot \hat{n})^2 = \sum_{k_1 k_1'} \sum_{k_2 k_2'} L_{k_1 k_1'} L_{k_2 k_2'} [u(k_1) u^*(k_1') + v(k_1)\, v^*(k_1')]$$

$$\times [u(k_2) u^*(k_2') + v(k_2)\, v^*(k_2')] (\alpha_{k_1}^\dagger\, \alpha_{k_1'} + \beta_{k_1}^\dagger\, \beta_{k_1'})(\alpha_{k_2}^\dagger\, \alpha_{k_2'} + \beta_{k_2}^\dagger\, \beta_{k_2'}) \,.$$

In the averaging of this expression, the terms $\underset{\sim}{k}_1 = \underset{\sim}{k}_1{}'$ and $\underset{\sim}{k}_2 = \underset{\sim}{k}_2{}'$ do not occur because the corresponding matrix elements vanish. The only nonzero terms are those involving four α or four β operators

$$\left\langle (\underset{\sim}{J} \cdot \hat{\underset{\sim}{n}})^2 \right\rangle = \sum_{\underset{\sim}{k}\,\underset{\sim}{k}'} L_{\underset{\sim}{k}\underset{\sim}{k}'} L_{\underset{\sim}{k}'\underset{\sim}{k}} \left| u(\underset{\sim}{k})\, u^*(\underset{\sim}{k}') + v(\underset{\sim}{k})\, v^*(\underset{\sim}{k}') \right|^2$$

$$\times \left[\left\langle \alpha^\dagger_{\underset{\sim}{k}}\, \alpha_{\underset{\sim}{k}} \right\rangle \left(1 - \left\langle \alpha^\dagger_{\underset{\sim}{k}'}\, \alpha_{\underset{\sim}{k}'} \right\rangle \right) + \left\langle \beta^\dagger_{\underset{\sim}{k}}\, \beta_{\underset{\sim}{k}} \right\rangle \left(1 - \left\langle \beta^\dagger_{\underset{\sim}{k}'}\, \beta_{\underset{\sim}{k}'} \right\rangle \right) \right] \tag{5.5}$$

According to Eq. (5.3), the only nonzero terms in this equation are for $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ differing only in their azimuthal angles ϕ and ϕ'. Since the quasi-particle transformation does not depend on the azimuthal angle, the u's and v's drop out completely [when Eq. (2.3) is used] and all the statistical factors are the same:

$$\left\langle (\underset{\sim}{J} \cdot \hat{\underset{\sim}{n}})^2 \right\rangle = 2 \sum_{\underset{\sim}{k}\,\underset{\sim}{k}'} f(\underset{\sim}{k}) \left[1 - f(\underset{\sim}{k}) \right] L_{\underset{\sim}{k}\underset{\sim}{k}'} L_{\underset{\sim}{k}'\underset{\sim}{k}} ,$$

or, using closure,

$$\left\langle (\underset{\sim}{J} \cdot \hat{\underset{\sim}{n}})^2 \right\rangle = 2 \sum_{\underset{\sim}{k}} f(\underset{\sim}{k}) \left[1 - f(\underset{\sim}{k}) \right] (L^2)_{\underset{\sim}{k}\underset{\sim}{k}} . \tag{5.6}$$

For the diagonal matrix element of L^2 appropriate to a cylindrical container, we have

$$(L^2)_{\underset{\sim}{k}\underset{\sim}{k}} = \frac{1}{2} (\underset{\sim}{k} \times \hat{\underset{\sim}{n}})^2 \left\langle x^2 + y^2 \right\rangle ,$$

where

$$\left\langle x^2 + y^2 \right\rangle = \frac{1}{V} \int d^3 r \, (x^2 + y^2) .$$

The moment of inertia is therefore

$$I(0) = \left\langle x^2 + y^2 \right\rangle 2\beta \sum_{\underset{\sim}{k}} \frac{1}{2} (\underset{\sim}{k} \times \underset{\sim}{\hat{n}})^2 f(\underset{\sim}{k}) [1 - f(\underset{\sim}{k})] . \tag{5.7}$$

For a spherically symmetric gap the angular average of $\frac{1}{2} (\underset{\sim}{k} \times \underset{\sim}{\hat{n}})^2$ is equal to the angular average of $(\underset{\sim}{k} \cdot \underset{\sim}{\hat{v}})^2$, which means

$$\frac{I(0)}{I_0} = \frac{M_f(0)}{mn} , \tag{5.8}$$

where I_0 is the rigid-body moment of inertia. For an ideal gas, therefore, we have $I(0) = I_0$.

This result for the spherically symmetric case has been obtained previously by more tedious methods.[17-18] The statistical approach employed here is more attractive because it emphasizes the role of the energy spectrum of the system. It is particularly easy to apply to quasi-particle models, which encompass a large class of approximations to the many-body problem.

For the asymmetric case $[E = E(k, \theta)]$, Eq. (5.7) may be transformed to

$$\frac{I(0)}{I_0} = K_2 , \tag{5.9}$$

where K_2 was defined by Eqs. (4.12) and (4.13). This result is easily understood by recalling that, for the case considered in this section, the flow velocity is always perpendicular to the quantization axis. Hence we expect that Eq. (5.8), originally written for the spherically symmetric case, should now be valid when we use Eq. (4.11) for $M_f(0)/nm$ with $\tau = \pi/2$.

It must be emphasized that the results of this paper are based on the quasiparticle approximation and that the interaction between quasiparticles has been ignored. These interactions may be important for the calculation of the moment of inertia,[17] but the investigation of their effect has not yet been completed. Similarly, the problem of viscosity has not been discussed. However, we do expect the viscosity to vanish at low temperatures in the limit of small flow velocities. This follows from the fact that in this limit only a very limited class of excitations are possible in view of the modified energy spectrum in the superfluid state. In any case the viscosity should be drastically reduced below the viscosity in the normal fluid which, in the limit $T \to 0$, varies as T^{-2}.[19]

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